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Partial Sums of Certain Analytic Functions

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Abstract. The object of the present paper is to consider of starlikeness and convexity of partial sums of certain analytic functions in the open unit disk

1 Introduction

Let A denote the class of functions $f(z)$ of the form

$$(1) \quad f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $S^*(\alpha)$ be the subclass of A consisting of functions $f(z)$ which satisfy

$$(2) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in U)$$

for some $\alpha (0 \leq \alpha < 1)$. A function $f(z)$ in $S^*(\alpha)$ is said to be starlike of order α in U . Furthermore, let $K(\alpha)$ denote the subclass of A consisting of all functions $f(z)$ which satisfy

$$(3) \quad \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (z \in U)$$

for some $\alpha (0 \leq \alpha < 1)$. A function $f(z)$ belonging to $K(\alpha)$ is said to be convex of order α in U . We note that $f(z) \in S^*(\alpha)$ if and only if $zf'(z) \in K(\alpha)$ and denote by $S^*(0) \equiv S^*$ and $K(0) \equiv K$. For $f(z) \in A$, we introduce the partial sum of $f(z)$ by

$$(4) \quad f_n(z) = z + \sum_{k=2}^n a_k z^k.$$

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Remark 1. It is well-known that

(i) $f(z) = \frac{z}{(1-z)^2} = z + \sum_{k=2}^{\infty} k z^k$ is the extremal function for the class S^* . But $f_2(z) = z + 2z^2 \notin S^*$.

(ii) $f(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k$ is the extremal function for the class K . But $f_2(z) = z + z^2 \notin K$.

For the partial sums $f_n(z)$ of $f(z) \in S^*$, Szegő [2] showed
([2])

Theorem 1. (i) $f(z) \in S^*$ implies that $f_n(z) \in S^*$ for $|z| < \frac{1}{4}$. The result is sharp.
(ii) $f(z) \in S^*$ implies that $f_n(z) \in K$ for $|z| < \frac{1}{8}$. The result is sharp.

Further, Padmanabhan [1] proved

Theorem 2. If $f(z)$ is 2-valently starlike in U , then $f_n(z)$ is 2-valently starlike for $|z| < \frac{1}{6}$. The result is sharp.

2 Function $F_n(z)$

Let us define the function $F_n(z)$ which is the partial sum of $f(z) \in A$ by

$$(5) \quad F_n(z) = z + a_n z^n.$$

Theorem 3. The function $F_n(z)$ satisfies

$$(6) \quad \frac{1 - n|a_n|r^{n-1}}{1 - |a_n|r^{n-1}} \leq \operatorname{Re} \left\{ \frac{zF'_n(z)}{F_n(z)} \right\} \leq \frac{1 + n|a_n|r^{n-1}}{1 + |a_n|r^{n-1}}$$

for $0 \leq r < \sqrt[n-1]{\frac{1}{|a_n|}} \leq 1$. Therefore $F_n(z) \in S^*(\alpha)$ for $0 \leq r < \sqrt[n-1]{\frac{1-\alpha}{(n-\alpha)|a_n|}} \leq 1$.

Proof. Note that

$$(7) \quad \frac{zF'_n(z)}{F_n(z)} = \frac{z + na_n z^n}{z + a_n z^n} = n - \frac{n-1}{1 + a_n z^{n-1}}.$$

It follows from (7) that

$$(8) \quad \operatorname{Re} \left\{ \frac{zF'_n(z)}{F_n(z)} \right\} = n - (n-1) \frac{1 + |a_n|r^{n-1}\cos\theta}{1 + |a_n|^2 r^{2(n-1)} + 2|a_n|r^{n-1}\cos\theta}.$$

Since the righthand side of (8) is increasing for $\cos\theta$ if $|a_n| < 1$, we obtain (6). Further, we also see that

$$(9) \quad \operatorname{Re} \left\{ \frac{zF'_n(z)}{F_n(z)} \right\} \geq \frac{1 - n|a_n|r^{n-1}}{1 - |a_n|r^{n-1}} > \alpha$$

for $0 \leq r < \sqrt[n-1]{\frac{1-\alpha}{(n-\alpha)|a_n|}} \leq 1$. This completes the proof of the theorem. □

Next we derive

Theorem 4. *The function $F_n(z)$ satisfies*

$$(10) \quad \frac{1 - n^2 |a_n| r^{n-1}}{1 - n |a_n| r^{n-1}} \leq \operatorname{Re} \left\{ 1 + \frac{z F_n''(z)}{F_n'(z)} \right\} \leq \frac{1 + n^2 |a_n| r^{n-1}}{1 + n |a_n| r^{n-1}}$$

for $0 \leq r < \sqrt[n-1]{\frac{1}{n|a_n|}} \leq 1$. Therefore, $F_n(z) \in K$ for $0 \leq r < \sqrt[n-1]{\frac{1-\alpha}{n(n-\alpha)|a_n|}} \leq 1$.

Proof. Noting that

$$(11) \quad 1 + \frac{z F_n''(z)}{F_n'(z)} = n - \frac{n-1}{1 + n a_n z^{n-1}},$$

we have

$$(12) \quad \operatorname{Re} \left\{ 1 + \frac{z F_n''(z)}{F_n'(z)} \right\} = n - (n-1) \frac{1 + n |a_n| r^{n-1} \cos \theta}{1 + n^2 |a_n|^2 r^{2(n-1)} + 2n |a_n| r^{n-1} \cos \theta},$$

which derives (10). □

By virtue of the above theorems, we have

Conjecture 1. *For the partial sum $f_n(z)$ of $f(z)$ belonging to the class A ,*

(i) $f_n(z) \in S^*(\alpha)$ for $0 \leq r < \sqrt[n-1]{\frac{1-\alpha}{(n-\alpha)|a_n|}} \leq 1$,

and

(ii) $f_n(z) \in K(\alpha)$ for $0 \leq r < \sqrt[n-1]{\frac{1-\alpha}{n(n-\alpha)|a_n|}} \leq 1$.

3 The partial sums of certain analytic functions

In this section, we consider the partial sums of functions $f(z) = \frac{z}{1-z}$ and $f(z) = \frac{z}{(1-z)^2}$.

Theorem 5. *Let $f_3(z) = z + z^2 + z^3$ be the partial sum of $f(z) = \frac{z}{1-z}$ which is the extremal function of the class K . Then $f_3(z) \in S^*(\frac{626}{961})$ for $0 \leq r < \beta$ ($\frac{1}{7} < \beta < \frac{1}{6}$), where β is the positive root of*

$$(13) \quad x^4 - 8x^3 + 9x^2 - 8x + 1 = 0 \quad (0 < x < \frac{1}{\sqrt{3}}).$$

Proof. We consider α such that

$$(14) \quad \operatorname{Re} \left\{ \frac{z f_3'(z)}{f_3(z)} \right\} = \operatorname{Re} \left\{ 3 - \frac{2+z}{1+z^2+z^3} \right\} > \alpha$$

for $0 \leq r < \beta$. This implies that

$$(15) \quad \operatorname{Re} \left\{ \frac{2+z}{1+z^2+z^3} \right\} = 1 + \frac{(1-r^2)(1+r^2+r\cos\theta)}{1-r^2+r^4+4r^2\cos^2\theta+2r(1+r^2)\cos\theta} < 3-\alpha,$$

that is, that

$$(16) \quad \operatorname{Re} \left\{ \frac{(1-r^2)(1+r^2+r\cos\theta)}{1-r^2+r^4+4r^2\cos^2\theta+2r(1+r^2)\cos\theta} \right\} < 2-\alpha.$$

Let the function $g(t)$ be given by

$$(17) \quad g(t) = \frac{(1-r^2)(1+r^2+rt)}{1-r^2+r^4+4r^2t^2+2r(1+r^2)t} \quad (t = \cos\theta).$$

Then we have

$$(18) \quad g'(t) = \frac{r(r+1)(r-1)(1+5r^2+r^4+4r^2t^2+8r(1+r^2)t)}{(1-r^2+r^4+4r^2t^2+2r(1+r^2)t)^2}.$$

Letting

$$(19) \quad h(t) = 1+5r^2+r^4+4r^2t^2+8r(1+r^2)t,$$

we see that

$$(i) h(t) < 0 \Rightarrow g'(t) > 0,$$

$$(ii) h(t) > 0 \Rightarrow g'(t) < 0,$$

and

$$(iii) h(t) = 0 \text{ for } t = \frac{-2(1+r^2) \pm \sqrt{3(1+r^2+r^4)}}{2r}.$$

If we write

$$t_1 = \frac{-2(1+r^2) + \sqrt{3(1+r^2+r^4)}}{2r} < 0,$$

then $0 \leq r \leq \beta$ implies that $t_1 \leq -1$, so that, $h(t) \geq 0$. This gives us that

$$(20) \quad g(t) \leq g(-1) = \frac{1-r+r^3-r^4}{1-2r+3r^2-2r^3+r^4} = \frac{g_1(r)}{g_2(r)},$$

It is easy to check that $g_1(r)$ is decreasing for r ($0 \leq r < \frac{1}{\sqrt{3}}$). Therefore

$$(21) \quad \frac{8-2\sqrt{3}}{9} = g_1\left(\frac{1}{\sqrt{3}}\right) < g_1(r) \leq g_1(0) = 1.$$

Also, $g_2(r)$ is decreasing for r ($0 \leq r < \beta$), because $g_2'(0) = -2 < 0$ and $g_2'(\frac{1}{6}) = -\frac{31}{27} < 0$. This gives that

$$(22) \quad \frac{961}{1296} = g_2\left(\frac{1}{6}\right) < g_2(r) \leq 1.$$

Consequently, we conclude that

$$(23) \quad g(t) \leq g(-1) = \frac{g_1(r)}{g_2(r)} < \frac{1296}{961} = 2 - \alpha,$$

that is, that $\alpha = \frac{626}{961} = 0.651 \dots$.

Thus we have that

$$(24) \quad \operatorname{Re} \left\{ \frac{zf'_3(z)}{f_3(z)} \right\} > \alpha \quad \left(\alpha = \frac{626}{961} \right)$$

for $0 \leq r < \beta$.

□

Finally, we obtain

Theorem 6. Let $f_3(z) = z + 2z^2 + 3z^3$ be the partial sum of the Keobe function $f(z) = \frac{z}{(1-z)^2}$ which is the extremal function for the class S^* . Then $f_3(z) \in K(\frac{3191}{15876})$ for $0 \leq r < \beta$ ($\frac{1}{14} < \beta < \frac{1}{13}$), where β is the positive root of

$$(25) \quad 81x^4 - 162x^3 + 72x^2 - 18x + 1 = 0 \quad (0 \leq x < \frac{1}{3}).$$

Proof. Since

$$(26) \quad \operatorname{Re} \left\{ 1 + \frac{zf''_3(z)}{f'_3(z)} \right\} = \operatorname{Re} \left\{ 3 - \frac{2(1+2z)}{1+4z+9z^2} \right\} > \alpha$$

implies that

$$(27) \quad \operatorname{Re} \left\{ \frac{1 * 2z}{1 + 4z + 9z^2} \right\} = \frac{1}{2} + \frac{4r(1-9r^2)\cos\theta * 1 - 81r^4}{2(1-2r^2+81r^4+8r(1+9r^2)\cos\theta+36r^2\cos^2\theta)} < \frac{3-\alpha}{2},$$

we have to check that

$$(28) \quad \frac{(1-9r^2)(1+9r^2+4r\cos\theta)}{1-2r^2+81r^4+8r(1+9r^2)\cos\theta+36r^2\cos^2\theta} < 2-\alpha.$$

If we let

$$(29) \quad h(t) = \frac{(1-9r^2)(1+9r^2+4rt)}{1-2r^2+81r^4+8r(1+9r^2)t+36r^2t^2},$$

then we have

$$h(t) \leq h(-1) = \frac{1-4r+36r^3-81r^4}{1-8r+34r^2-72r^3+81r^4} \equiv \frac{g_1(r)}{g_2(r)}.$$

Noting that $0 < g_1(r) < 1$, and $g_2(r) > g_2(\frac{1}{13}) = \frac{15876}{28561}$, we have

$$h(t) \leq h(-1) < \frac{1}{g_2(r)} < \frac{28561}{15876} = 2 - \alpha,$$

which implies that $\alpha = \frac{3191}{15876} = 0.200 \dots$.

□

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